Topological Entropy and pressure.

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Let φ be a continuous function a topological space X, $T: X \circlearrowleft$ - continuous. Define
$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(T^k x)$$
- sum of φ along a trajectory.

Now let $X = C_A$ for an aperiodic A, $T = T_6$.

Let $\varphi_C := \sup\{\varphi(x) : x \in C\}$ for a cylinder C.

Thm. $P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{C \in \mathcal{A}_n} e^{(S_n\varphi)_C}\right)$ exists.

It is called *the pressure of φ*.

Pf. Let
$$u_n := \sum_{C\in\mathcal{A}_n} e^{(S_n\varphi)_C}. \text{ Then}$$
$$u_{n+m} = \sum_{\substack{A\in\mathcal{A}_n \\ B\in\mathcal{A}_m}} e^{(S_{n+m}\varphi)_{AB}} \le \sum_{\substack{A\in\mathcal{A}_n \\ B\in\mathcal{A}_m}} e^{(S_n\varphi)_A + (S_m\varphi)_B} =$$
$$\left(\sum_{A\in\mathcal{A}_n} e^{(S_n\varphi)_A}\right)\left(\sum_{B\in\mathcal{A}_m} e^{(S_m\varphi)_B}\right) = u_n u_m. \text{ so}$$
$\log u_{n+m} \le \log u_n + \log u_m$. So $P(\varphi) = \lim \frac{1}{n} u_n$ exists ∎

In the special case $\varphi = 0$, $P(\varphi)$ is called *a topological entropy of (X,T)*. For the full shift,
$h_{top}(T) = \lim \frac{1}{n} \log 6^n = \log 6$.
For a subshift defined by an A,
$h_{top}(T_A) = \lim \frac{1}{n} \log(\#\{C \in \mathcal{A}_n : C \cap C_A \neq \emptyset\}) = \lim \frac{1}{n} N_n^l(C_A) = \log r(A)$.
We already made this calculation! ($r(A)$ is the spectral radius of A)

Thm (The first half of the variational principle).
$$P(\varphi) \ge \sup_{\substack{\mu\text{-probability} \\ T\text{-invariant}}} \left(h_\mu(T) + \int_X \varphi \, d\mu\right).$$

Remark There is equality, reached exactly for Gibbs measures at least for "nice" φ. In particular,
$$\boxed{h_{top} = \sup_{\mu\text{-}T\text{ invariant}} h_\mu(T)}$$

The proof is based on the following inequality:

Let $p_1 + \dots + p_n = 1$, $p_j \ge 0$. $a_i \in \mathbb{R}$, $i \le n$. Then
$$\sum p_i(a_i - \log p_i) \le \log \sum_{j=1}^n e^{a_j}.$$
Equality $\Leftrightarrow$ $p_i = \frac{e^{a_i}}{\sum e^{a_j}}$

Proven by Lagrange multipliers/induction on n.

Pf of the variational principle.

Let us index p by $A \in \mathcal{A}_n$.

$p_A := \mu(A)$, $\quad a_A = (S_n \varphi)_A$. Then

$$\sum \mu(A)\left((S_n\varphi)_A - \log \mu(A)\right) \le \log \sum_{A \in \mathcal{A}_n} e^{(S_n \varphi)_A}$$

$\Rightarrow$

$$\frac{h_n(\mu)}{n} + \frac{1}{n}\sum_{A \in \mathcal{A}} \mu(A)(S_n\varphi)_A \le \frac{1}{n} \log \sum e^{(S_n\varphi)_A}$$

$\text{RHS} \xrightarrow[n\to\infty]{} P(\varphi)$ (by def.)

$\frac{h_n(\mu)}{n} \to h(\mu)$.

$$\sum \mu(A)(S_n\varphi)_A \ge \int_{X_b} (S_n\varphi)\, d\mu \overset{\mu \text{ is } T \text{ invariant}}{=} n \int_{X_b} \varphi \, d\mu$$

this was max